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# The Multivariate Gram-Charlier Series Applied to Random Signal Detection

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This report comprises two parts. The first part focuses on extending the well-known Gram-Charlier series expansion technique for univariate probability density functions to multivariate functions. The approach employed here is particularly attractive because it avoids explicit use of tensor analysis and multivariate Hermite polynomials. The desired expansion is accomplished via straightforward application of Kronecker products and matrix calculus formulae. The second part of this report addresses the problem of signal detection in the presence of background noise. A canonical representation of the likelihood ratio is derived that applies to a broad class of multivariate random signals embedded in Gaussian noise. The representation takes the form of an infinite series whose terms depend on the received measurement vector and the signal and noise statistics. It prescribes a solution to the binary detection problem in a variety of sonar applications.

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# THE MULTIVARIATE GRAM-CHARLIER SERIES APPLIED TO RANDOM SIGNAL DETECTION

## 1. INTRODUCTION

The application of probability theory and statistical modeling to the analysis of physical systems frequently leads to mathematical problems that are not amenable to closed-form solution. Complicated nonlinearities, unwieldy computations, and other manipulative difficulties often necessitate the use of numerical techniques and/or approximations. In the latter case, several alternatives are possible. One common approach is to represent the probability density function (PDF) as an infinite series of orthogonal polynomials, with the expansion coefficients being specified in terms of statistical moments.[1,2,3] The key to successful application of this procedure lies in judiciously selecting the orthogonal polynomials. For practical applications, series approximations prove useful only when a small number of terms suffice to provide the required accuracy. In this context, the Gram-Charlier series[3] is known to render good service, especially in situations where the original PDF exhibits Gaussian characteristics; viz., unimodality, continuity, and bounded variation. The theoretical and computational aspects of approximating a univariate PDF via the Gram-Charlier series have been extensively documented in the literature.[1-5] Unfortunately, corresponding results for a multivariate PDF are conspicuously meager. This is due to the added complexity that typically characterizes $n$-dimensional mathematical analysis, as well as a general lack of familiarity with the properties of orthogonal polynomials in several variables. Such considerations need not be of great concern, however, since the Gram-Charlier series expansion technique can actually be extended to higher dimensions without explicit use of complicated analytical procedures or functions; viz., tensor analysis[6] and multivariate orthogonal polynomials.[7] A straightforward extension is described in the first part of this report.

Section 2.1 focuses on developing a suitable infinite series representation for the $n$-dimensional Dirac delta function. The Gram-Charlier expansion of an arbitrary multivariate PDF is then obtained as demonstrated in section 2.2. There, analytical expressions for the unknown expansion coefficients are also developed and explicitly evaluated in terms of the central moments of the distribution. In section 2.3, a linear integral equation of the second kind is derived for subsequent use in solving an important binary detection problem that arises in sonar array processing applications. Finally, section 2.4 provides a comprehensive discussion of the mathematical results.

The second part of this report examines the problem of detecting random multivariate signals embedded in Gaussian noise. A canonical representation for the likelihood ratio is derived in the form of an infinite series whose terms depend on the received measurement vector, central moments of the random signal, and the background noise statistics. The representation explicitly prescribes the optimum processing required to detect a broad class of multivariate random signals. Unfortunately, few representations of the type developed here are available in the literature. Those that do appear usually avoid detailed characterization of non-Gaussian signals. Low signal-to-noise ratio conditions are typically assumed to exist, allowing the use of simplifying

approximations that preclude explicit specification of the signal PDF; only its mean and covariance are required, as in the case of locally optimal Bayes detectors.[8,9] A notable exception to the aforementioned approach is the work of Kelly and Tague.[10] There, a canonical representation is developed for non-Gaussian univariate signals without any restrictions on input signal-to-noise ratio. The only assumptions made are that the background noise is Gaussian distributed and the signal PDF can be represented by a Gram-Charlier series expansion. The first assumption is reasonable for most sonar signal processing applications, since ambient noise in the ocean environment usually exhibits Gaussian statistical characteristics.[11] The second assumption imposes no significant restrictions either, since it is satisfied by a wide variety of acoustic signals encountered in practice. In fact, the generality of Kelly and Tague's work renders it an important contribution to the signal processing literature.

Despite its obvious attributes, the aforementioned representation was developed for univariate signals only; thus it cannot be used effectively for array processing applications involving vector-valued measurements. A desire to overcome this limitation provides impetus for the present study. Following the approach originally employed by Kelly and Tague,[10] pertinent mathematical formulae are developed, extended, and modified to accommodate multivariate signals. The resulting $n$-dimensional representation retains all the functional characteristics, attributes, and generality of its one-dimensional counterpart, and as expected, becomes identical to it for the case $n = 1$. An attractive feature of all formulae is that statistical means and covariances associated with signal and noise always appear in their compact vector and matrix formats, respectively. The need to explicitly depict individual vector components or matrix elements is avoided via utilization of Kronecker products.[12] Indeed, this artifice greatly simplifies expressions that would otherwise be cumbersome and not amenable to analysis and/or interpretation.

Section 3.1 provides a mathematical formulation of the binary detection problem for random multivariate signals embedded in Gaussian background noise. The problem is examined in the context of sonar array signal processing, and the solution is specified as a statistical hypothesis test; viz., the "likelihood ratio test."[2,10,13] In section 3.2, a canonical representation for the likelihood ratio is developed. This representation takes the form of an infinite series whose terms depend on the received measurement vector together with the signal and noise statistics. Some interesting possibilities for approximating the likelihood ratio are discussed in section 3.3. The various alternatives are revealed via analysis of the series expansion. In addition, insight is gained regarding the optimal signal processing required to detect non-Gaussian signals embedded in Gaussian noise.

It is not necessary to go through the mathematical derivations presented in the first part of this report completely before proceeding to the signal detection material discussed in the second part. The reader interested primarily in results and applications may omit the first part entirely, or content himself with making only slight acquaintance with the multivariate Gram-Charlier series derivation. Pertinent results used in the second part are appropriately referenced to the first part as they occur.

## 2. THE MULTIVARIATE GRAM-CHARLIER SERIES

### 2.1 REPRESENTATION OF THE *n*-DIMENSIONAL DIRAC DELTA FUNCTION

It is well known[14] that the functions

$$P_k(z) = \left(2^k \sqrt{\pi}\, k!\right)^{-1/2} H_k(z) \qquad k = 0,1,2,\ldots, \tag{1}$$

where $H_k(z)$ are Hermite polynomials defined by the formulae[14]

$$H_k(z) = (-1)^k e^{z^2} \frac{d^k e^{-z^2}}{dz^k} \qquad k = 0,1,2,\ldots, \tag{2}$$

form an orthonormal system on the interval $-\infty < z < \infty$. Accordingly, these functions must satisfy Bessel's equality,[15] or the equivalent completeness relation

$$\delta(\tilde{z} - z) = \sum_{k=0}^{\infty} P_k(\tilde{z}) P_k(z) = \frac{e^{\left(\tilde{z}^2 + z^2\right)}}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{2^{-k}}{k!} \left(\frac{\partial^2}{\partial \tilde{z}\, \partial z}\right)^k \left(e^{-\left(\tilde{z}^2 + z^2\right)}\right), \tag{3}$$

where $\delta(\tilde{z} - z)$ represents the one-dimensional Dirac delta function.[15]

If one makes the substitutions*

$$u = u_i / \sqrt{2} \qquad \text{for } u = (\tilde{z}, z), \tag{4}$$

in equation (3), and uses the chain rule formula for derivatives

$$\frac{\partial}{\partial u} = \frac{\partial u_i}{\partial u} \frac{\partial}{\partial u_i} = \sqrt{2} \frac{\partial}{\partial u_i} \qquad \text{for } u = (\tilde{z}, z), \tag{5}$$

along with the relation[15]

$$\delta\left(\frac{\tilde{z}_i - z_i}{\sqrt{2}}\right) = \sqrt{2}\, \delta(\tilde{z}_i - z_i), \tag{6}$$

---

* Throughout this report, the notation employed in equation (4) will be used. The desired substitutions are $\tilde{z} = \tilde{z}_i / \sqrt{2}$ and $z = z_i / \sqrt{2}$. To obtain these fomulae, simply replace the letter u with $\tilde{z}$ and z, respectively.

it is readily shown that

$$\delta(\tilde{z}_i - z_i) = \frac{e^{1/4\left(\tilde{z}_i^2 + z_i^2\right)}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\partial^2}{\partial\tilde{z}_i \partial z_i}\right)^k e^{-1/2\left(\tilde{z}_i^2 + z_i^2\right)}. \tag{7}$$

For notational convenience, one can introduce the scalar function

$$\phi(u_i) = \frac{e^{-1/2(u_i^2)}}{\sqrt{2\pi}} \qquad \text{for } u = (\tilde{z}, z), \tag{8}$$

and the differential operator

$$L_i = \frac{\partial^2}{\partial\tilde{z}_i \partial z_i}\ . \tag{9}$$

These quantities allow equation (7) to be recast in a more compact form:

$$\sqrt{\phi(\tilde{z}_i)\phi(z_i)}\ \delta(\tilde{z}_i - z_i) = \sum_{k=0}^{\infty} \frac{1}{k!}(L_i)^k \phi(\tilde{z}_i)\phi(z_i). \tag{10}$$

Although equations (3) and (10) describe identical completeness relations in one dimension, the latter expression is more readily extended to $n$-dimensions. To this end, let

$$\underline{u} = [u_1, u_2, \ldots, u_n]^T \qquad \text{for } u = (\tilde{z}, z), \tag{11}$$

where the superscript "$T$" denotes matrix transpose, and observe that

$$\prod_{i=1}^{n} \phi(u_i) = \Phi(\underline{u}) = \frac{e^{-1/2(\underline{u}^T \underline{u})}}{(2\pi)^{n/2}} \qquad \text{for } u = (\tilde{z}, z), \tag{12}$$

$$\prod_{i=1}^{n} \delta(\tilde{z}_i - z_i) = \delta(\underline{\tilde{z}} - \underline{z}). \tag{13}$$

Using the preceding formulae in conjunction with equation (10) leads to the result

$$\sqrt{\Phi(\underline{\tilde{z}})\Phi(\underline{z})}\ \delta(\underline{\tilde{z}} - \underline{z}) = \left[\prod_{i=1}^{n}\left(\sum_{k_i=0}^{\infty} \frac{(L_i)^{k_i}}{k_i!}\right)\right]\Phi(\underline{\tilde{z}})\Phi(\underline{z}). \tag{14}$$

The product of $n$ infinite series appearing in square brackets on the right-hand side of equation (14) can be expressed as a single infinite series by repeated application of the double summation formula[16]

$$\left(\sum_{k=0}^{\infty} A_k\right)\left(\sum_{m=0}^{\infty} B_m\right) = \sum_{m=0}^{\infty}\sum_{k=0}^{m} A_k B_{m-k}. \tag{15}$$

To see this, let

$$I_p = \prod_{i=p}^{n}\left(\sum_{k_i=0}^{\infty}\frac{(L_i)^{k_i}}{k_i\,!}\right) \qquad p = 1,2,\ldots,n. \tag{16}$$

Equation (14) then takes the form

$$\sqrt{\Phi(\underline{\tilde{z}})\Phi(\underline{z})}\,\delta(\underline{\tilde{z}} - \underline{z}) = I_1\Phi(\underline{\tilde{z}})\Phi(\underline{z}). \tag{17}$$

It now follows from equation (15), equation set (16), and the Binomial Expansion Theorem[16] that

$$\begin{aligned}
I_1 &= \left(\sum_{k_1=0}^{\infty}\frac{(L_1)^{k_1}}{k_1\,!}\right)\left(\sum_{k_2=0}^{\infty}\frac{(L_2)}{k_2\,!}\right)I_3 \\
&= \left(\sum_{k_2=0}^{\infty}\sum_{k_1=0}^{k_2}\frac{(\mathrm{L}_1)^{k_1}(L_2)^{k_2-k_1}}{k_1\,!(k_2-k_1)!}\right)I_3 \\
&= \left(\sum_{k_2=0}^{\infty}\frac{1}{\mathrm{k}_2\,!}\sum_{k_1=0}^{k_2}\frac{k_2\,!}{k_1\,!(k_2-k_1)!}(L_1)^{k_1}(L_2)^{k_2-k_1}\right)I_3 \\
&= \left(\sum_{k_2=0}^{\infty}\frac{1}{k_2\,!}(L_1+L_2)^{k_2}\right)I_3\,.
\end{aligned} \tag{18}$$

Repeating this procedure yields a similar expression

$$
\begin{aligned}
I_1 &= \left(\sum_{k_2=0}^{\infty} \frac{1}{k_2\,!}(L_1+L_2)^{k_2}\right)\left(\sum_{k_3=0}^{\infty} \frac{(L_3)^{k_3}}{k_3\,!}\right) I_4 \\
&= \left(\sum_{k_3=0}^{\infty} \sum_{k_2=0}^{k_3} \frac{(L_1+L_2)^{k_2}(L_3)^{k_3-k_2}}{k_2\,!(k_3-k_2)!}\right) I_4 \\
&= \left(\sum_{k_3=0}^{\infty} \frac{1}{k_3\,!} \sum_{k_2=0}^{k_3} \frac{k_3\,!}{k_2\,!(k_3-k_2)!}(L_1+L_2)^{k_2}(L_3)^{k_3-k_2}\right) I_4 \\
&= \left(\sum_{k_3=0}^{\infty} \frac{1}{k_3\,!}(L_1+L_2+L_3)^{k_3}\right) I_4,
\end{aligned}
\tag{19}
$$

and, finally, by induction the following result is obtained:

$$
I_1 = \sum_{k_h=0}^{\infty} \frac{1}{k_n\,!}\left(\sum_{i=1}^{n} L_i\right)^{k_n}. \tag{20}
$$

The desired single series representation for $I_1$ is now deduced by noting that

$$
\sum_{i=1}^{n} L_i = \sum_{i=1}^{n} \frac{\partial^2}{\partial \tilde{z}_i \partial z_i} = \left(\frac{\partial}{\partial \underline{\tilde{z}}}\right)^T \left(\frac{\partial}{\partial \underline{z}}\right), \tag{21}
$$

where

$$
\frac{\partial}{\partial \underline{u}} = \left[\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \ldots, \frac{\partial}{\partial u_n}\right]^T \qquad \text{for } u = (\tilde{z}, z), \tag{22}
$$

is the gradient operator associated with the vector $\underline{u}$. Combining the results embodied in equations (17), (20), and (21) yields the $n$-dimensional analogue of equation (10):

$$
\sqrt{\Phi(\underline{\tilde{z}})\Phi(\underline{z})}\,\delta(\underline{\tilde{z}} - \underline{z}) = \sum_{k=0}^{\infty} \frac{1}{k\,!}\left[\left(\frac{\partial}{\partial \underline{\tilde{z}}}\right)^T \left(\frac{\partial}{\partial \underline{z}}\right)\right]^k \Phi(\underline{\tilde{z}})\Phi(\underline{z}), \tag{23}
$$

or, equivalently,

$$\frac{\delta(\underline{\tilde{z}}-\underline{z})}{\sqrt{\Phi(\underline{\tilde{z}})\Phi(\underline{z})}}=\sum_{k=0}^{\infty}\frac{1}{k!}\frac{\left[\left(\frac{\partial}{\partial\underline{\tilde{z}}}\right)^{T}\left(\frac{\partial}{\partial\underline{z}}\right)\right]^{k}\Phi(\underline{\tilde{z}})\Phi(\underline{z})}{\Phi(\underline{\tilde{z}})\Phi(\underline{z})}. \tag{24}$$

To facilitate the Gram-Charlier expansion of an arbitrary multivariate PDF later on, it is convenient at this point to change the variables from $\underline{\tilde{z}}$ and $\underline{z}$ to $\underline{\tilde{x}}$ and $\underline{x}$ via the relations

$$\underline{\tilde{z}}=\Gamma^{1/2}(\underline{\tilde{x}}-\underline{\hat{x}}), \tag{25}$$

and

$$\underline{z}=\Gamma^{1/2}(\underline{x}-\underline{\hat{x}}), \tag{26}$$

where $\Gamma$ is an arbitrary ($n$ x $n$) symmetric, positive definite matrix, and $\underline{\hat{x}}$ is an arbitrary ($n$ x 1) vector (these quantities will be specified later). Substituting equations (25) and (26) into equation (24), and using the following relationships

$$\delta\left(\Gamma^{1/2}\{\underline{\tilde{x}}-\underline{x}\}\right)=\sqrt{|\Gamma|}\,\delta(\underline{\tilde{x}}-\underline{x}), \tag{27}$$

$$\frac{\partial}{\partial\underline{\tilde{z}}}=\Gamma^{1/2}\frac{\partial}{\partial\underline{\tilde{x}}}, \tag{28}$$

$$\frac{\partial}{\partial\underline{z}}=\Gamma^{1/2}\frac{\partial}{\partial\underline{x}}, \tag{29}$$

$$\Gamma=\Gamma^{1/2}\Gamma^{1/2}, \tag{30}$$

$$\left(\Gamma^{1/2}\right)^{T}=\Gamma^{1/2}, \tag{31}$$

leads to the expression

$$\frac{\delta(\underline{\tilde{x}}-\underline{x})}{N(\underline{\tilde{x}};\underline{\hat{x}},\Gamma)N(\underline{x};\underline{\hat{x}},\Gamma)}=\sum_{k=0}^{\infty}\frac{1}{k!}\frac{\left[\left(\frac{\partial}{\partial\underline{\tilde{x}}}\right)^{T}\Gamma\left(\frac{\partial}{\partial\underline{x}}\right)\right]^{k}N(\underline{\tilde{x}};\underline{\hat{x}},\Gamma)N(\underline{x};\underline{\hat{x}},\Gamma)}{N(\underline{\tilde{x}};\underline{\hat{x}},\Gamma)N(\underline{x};\underline{\hat{x}},\Gamma)}, \tag{32}$$

where

$$N(\underline{u};\hat{\underline{x}},\Gamma)=\frac{\Phi\left(\Gamma^{-1/2}\{\underline{u}-\hat{\underline{x}}\}\right)}{\sqrt{|\Gamma|}}=\frac{e^{-1/2(\underline{u}-\hat{\underline{x}})^{T}\Gamma^{-1}(\underline{u}-\hat{\underline{x}})}}{(2\pi)^{n/2}\sqrt{|\Gamma|}} \qquad \text{for } u=(\tilde{x},x), \tag{33}$$

is the $n$-dimensional Gaussian PDF.

Although the series representation for $\delta(\tilde{\underline{x}}-\underline{x})$ given by equation (32) is pleasingly compact, it does not yet exhibit the "separation attribute" necessary for utilitarian applications. Specifically, individual terms on the right-hand side of equation (32) are not written in a form that explicitly uncouples the $\tilde{\underline{x}}$ dependence from the $\underline{x}$ dependence. This can be accomplished, however, by using the Kronecker product formula for differential operators derived in appendix A (see equation (A-17)); viz.,

$$\left[\left(\frac{\partial}{\partial\tilde{\underline{x}}}\right)^{T}\Gamma\left(\frac{\partial}{\partial\underline{x}}\right)\right]^{k}=\left[\left(\Gamma^{1/2}\frac{\partial}{\partial\tilde{\underline{x}}}\right)^{T}\otimes\ldots\right]^{k}\left[\left(\Gamma^{1/2}\frac{\partial}{\partial\underline{x}}\right)\otimes\ldots\right]^{k} \qquad k=0,1,2,\ldots, \tag{34}$$

where the notation

$$[A\otimes\ldots]^{k}=\underbrace{A\otimes A\otimes\ldots\otimes A}_{k\text{ times}} \qquad k=0,1,2,\ldots, \tag{35}$$

has been employed. Substituting equation set (34) into equation (32) yields the desired result

$$\frac{\delta(\tilde{\underline{x}}-\underline{x})}{N(\tilde{\underline{x}};\hat{\underline{x}},\Gamma)N(\underline{x};\hat{\underline{x}},\Gamma)}=\sum_{k=0}^{\infty}\frac{1}{k!}\underline{\xi}_{k}^{T}(\tilde{\underline{x}};\hat{\underline{x}},\Gamma)\underline{\xi}_{k}(\underline{x};\hat{\underline{x}},\Gamma). \tag{36}$$

Here,

$$\underline{\xi}_{k}(\underline{u};\hat{\underline{x}},\Gamma)=\frac{1}{N(\underline{u};\hat{\underline{x}},\Gamma)}\left[\left(\Gamma^{1/2}\frac{\partial}{\partial\underline{u}}\right)\otimes\ldots\right]^{k}N(\underline{u};\hat{\underline{x}},\Gamma)$$

$$k=0,1,2,\ldots, \qquad \text{for } u=(\tilde{x},x), \tag{37}$$

is a ($n^k$ x 1) vector valued function of its argument $\underline{u}$. An alternative representation for $\underline{\xi}_{k}(\underline{u};\hat{\underline{x}},\Gamma)$, which is simpler to evaluate explicitly, may also be derived by combining equations (12), and (25), through (31) with equations (33) and equation set (37). Performing the necessary algebraic manipulations yields

$$\underline{\xi}_{k}(\underline{u};\hat{\underline{x}},\Gamma)=\underline{\psi}_{k}\left(\Gamma^{-1/2}\{\underline{u}-\hat{\underline{x}}\}\right) \qquad k=0,1,2,\ldots \qquad \text{for } u=(\tilde{x},x), \tag{38}$$

where

$$\underline{\psi}_k(\underline{u}) = \frac{1}{\Phi(\underline{u})}\left[\left(\frac{\partial}{\partial \underline{u}}\right)\otimes\ldots\right]^k \Phi(\underline{u}) \qquad k = 0,1,2,\ldots \quad . \tag{39}$$

## 2.2 EXPANSION OF AN ARBITRARY MULTIVARIATE PDF

Let $\underline{x}$ be a random ($n$ x 1) vector defined over all $n$-dimensional space $R_n$ whose PDF is given by $p(\underline{x})$. The mean value of $\underline{x}$ and its covariance matrix then satisfy the relations

$$\hat{\underline{x}} = \int_{R_n} \underline{x} p(\underline{x}) d\underline{x}, \tag{40}$$

$$\Gamma = \int_{R_n} (\underline{x} - \hat{\underline{x}})(\underline{x} - \hat{\underline{x}})^T p(\underline{x}) d\underline{x}. \tag{41}$$

Substituting these parameters into equation (36), multiplying both sides of that expression by $p(\tilde{\underline{x}})$, and then integrating the result over $R_n$ (here $\tilde{\underline{x}}$ is used as the variable of integration) yields the formula

$$p(\underline{x}) = N(\underline{x}; \hat{\underline{x}}, \Gamma) \sum_{k=0}^{\infty} \frac{1}{k!} \underline{\beta}_k^{\,T} \underline{\xi}_k(\underline{x}; \hat{\underline{x}}, \Gamma), \tag{42}$$

where

$$\underline{\beta}_k = \int_{R_n} \underline{\xi}_k(\tilde{\underline{x}}; \hat{\underline{x}}, \Gamma) p(\tilde{\underline{x}}) d\tilde{\underline{x}}. \tag{43}$$

Equation (42) describes the Gram-Charlier series expansion of an arbitrary multivariate PDF. The expansion coefficients $\underline{\beta}_k$ are specified in terms of the central moments of the PDF via equation (43). To compute $\underline{\beta}_k$ explicitly, it is first necessary to determine the vector-valued polynominals* $\underline{\xi}_k(\underline{x}; \hat{\underline{x}}, \Gamma)$ in closed form. To this end, recall that $\underline{\xi}_k(\underline{x}; \hat{\underline{x}}, \Gamma)$ is simply related to $\underline{\psi}_k(\Gamma^{-1/2}\{\underline{x} - \hat{\underline{x}}\})$ as is indicated by equation set (38). Consequently, knowledge of the latter provides sufficient information to evaluate the former. Since $\underline{\psi}_k(\underline{u})$ satisfies equation set (39), it immediately follows that

* The components of $\underline{\xi}_k(\underline{x}; \hat{\underline{x}}, \Gamma)$ are related to the Hermite polynomials in several variables described by Erdelyi.[7]

$$
\begin{aligned}
\underline{\psi}_{k+1}(\underline{u}) &= \frac{1}{\Phi(\underline{u})}\frac{\partial}{\partial \underline{u}}\otimes\left[\left(\frac{\partial}{\partial \underline{u}}\right)\otimes\ldots\right]^{k}\Phi(\underline{u}) \\
&= \frac{1}{\Phi(\underline{u})}\frac{\partial}{\partial \underline{u}}\otimes\left[\Phi(\underline{u})\underline{\psi}_{k}(\underline{u})\right] \\
&= \frac{1}{\Phi(\underline{u})}\frac{\partial\Phi(\underline{u})}{\partial \underline{u}}\otimes\underline{\psi}_{k}(\underline{u})+\frac{\partial}{\partial \underline{u}}\otimes\underline{\psi}_{k}(\underline{u}) \\
&= -\underline{u}\otimes\underline{\psi}_{k}(\underline{u})+\frac{\partial}{\partial \underline{u}}\otimes\underline{\psi}_{k}(\underline{u}).
\end{aligned}
\tag{44}
$$

Using the functional definition embodied in equation set (39), the recursion formula specified by equation (44), and the differentation rules given in appendix A, it is possible to determine polynomial representations for all members of the set $\{\underline{\psi}_{k}(\underline{u}): k = 0,1,2,\ldots\}$. Note, however, that $\underline{\psi}_{k}(\underline{u})$ is a ($n^k$ x 1) vector valued function of its argument $\underline{u}$. Unfortunately, since the dimensionality of $\underline{\psi}_{k}(\underline{u})$ grows exponentially with increasing index $k$, there is a concomitant increase in computational complexity associated with the explicit evaluation of each successive vector. To forestall this encumbrance, an alternative recursion formula may be employed. In particular, observe that (see appendix A)

$$
\frac{1}{\Phi(\underline{u})}\left[\left(\frac{\partial}{\partial \underline{u}}\right)\otimes\ldots\right]^{2k}\Phi(\underline{u}) = vec\left\{\frac{1}{\Phi(\underline{u})}\left[\left(\frac{\partial}{\partial \underline{u}}\left(\frac{\partial}{\partial \underline{u}}\right)^{T}\right)\otimes\ldots\right]^{k}\Phi(\underline{u})\right\}, \tag{45}
$$

and

$$
\frac{1}{\Phi(\underline{u})}\left[\left(\frac{\partial}{\partial \underline{u}}\right)\otimes\ldots\right]^{2k+1}\Phi(\underline{u}) = vec\left\{\frac{1}{\Phi(\underline{u})}\frac{\partial}{\partial \underline{u}}\otimes\left[\left[\left(\frac{\partial}{\partial \underline{u}}\left(\frac{\partial}{\partial \underline{u}}\right)^{T}\right)\otimes\ldots\right]^{k}\Phi(\underline{u})\right]\right\}, \tag{46}
$$

where

$$
\frac{\partial}{\partial \underline{u}}\left(\frac{\partial}{\partial \underline{u}}\right)^{T} = \left[\frac{\partial^2}{\partial u_i \partial u_j}\right] \qquad i,j = 1,2,\ldots,n\,, \tag{47}
$$

is a $(n \times n)$ matrix operator whose $(i,j)$ element is $\frac{\partial^2}{\partial u_i \partial u_j}$. This matrix is sometimes referred to in the literature[17] as the "Hessian" operator. If

$$M_{2k}(\underline{u}) = \frac{1}{\Phi(\underline{u})}\left[\left(\frac{\partial}{\partial \underline{u}}\left(\frac{\partial}{\partial \underline{u}}\right)^T\right)\otimes\ldots\right]^k \Phi(\underline{u}), \tag{48}$$

and

$$M_{2k+1}(\underline{u}) = \frac{1}{\Phi(\underline{u})}\frac{\partial}{\partial \underline{u}}\otimes\left[\left[\left(\frac{\partial}{\partial \underline{u}}\left(\frac{\partial}{\partial \underline{u}}\right)^T\right)\otimes\ldots\right]^k \Phi(\underline{u})\right], \tag{49}$$

it then follows from equation set (39) and equations (45), (46), (48), and (49) that

$$\underline{\psi}_k(\underline{u}) = vec\{M_k(\underline{u})\} \qquad k = 0,1,2,\ldots \tag{50}$$

A recursive relation for $M_{2k+1}(\underline{u})$ derives by substituting equation (48) into equation (49). The result is

$$\begin{aligned} M_{2k+1}(\underline{u}) &= \frac{1}{\Phi(\underline{u})}\frac{\partial}{\partial \underline{u}}\otimes\left[\Phi(\underline{u})M_{2k}(\underline{u})\right] \\ &= \frac{1}{\Phi(\underline{u})}\frac{\partial\Phi(\underline{u})}{\partial \underline{u}}\otimes M_{2k}(\underline{u}) + \frac{\partial}{\partial \underline{u}}\otimes M_{2k}(\underline{u}) \\ &= -\underline{u}\otimes M_{2k}(\underline{u}) + \frac{\partial}{\partial \underline{u}}\otimes M_{2k}(\underline{u}). \end{aligned} \tag{51}$$

Next, the matrix transpose of equation (49) is taken and the Hessian operator symmetry property

$$\left[\frac{\partial}{\partial \underline{u}}\left(\frac{\partial}{\partial \underline{u}}\right)^T\right]^T = \frac{\partial}{\partial \underline{u}}\left(\frac{\partial}{\partial \underline{u}}\right)^T \tag{52}$$

is used to obtain the expression

$$\Phi(\underline{u})M^{T}{}_{2k+1}(\underline{u})=\left(\frac{\partial}{\partial\underline{u}}\right)^{T}\otimes\left[\left[\left(\frac{\partial}{\partial\underline{u}}\left(\frac{\partial}{\partial\underline{u}}\right)^{T}\right)\otimes\ldots\right]^{k}\Phi(\underline{u})\right]. \tag{53}$$

Differentation of equation (53) subsequently yields the result

$$\frac{\partial}{\partial\underline{u}}\otimes\left[\Phi(\underline{u})M^{T}{}_{2k+1}(\underline{u})\right]=\left[\frac{\partial}{\partial\underline{u}}\left(\frac{\partial}{\partial\underline{u}}\right)^{T}\right]\otimes\left[\left[\left(\frac{\partial}{\partial\underline{u}}\left(\frac{\partial}{\partial\underline{u}}\right)^{T}\right)\otimes\ldots\right]^{k}\Phi(\underline{u})\right]$$

$$=\left[\left(\frac{\partial}{\partial\underline{u}}\left(\frac{\partial}{\partial\underline{u}}\right)^{T}\right)\otimes\ldots\right]^{k+1}\Phi(\underline{u}) \tag{54}$$

$$=\Phi(\underline{u})M_{2k+2}(\underline{u}),$$

from which it follows that

$$M_{2k+2}(\underline{u})=\frac{1}{\Phi(\underline{u})}\frac{\partial}{\partial\underline{u}}\otimes\left[\Phi(\underline{u})M^{T}_{2k+1}(\underline{u})\right]$$

$$=\frac{1}{\Phi(\underline{u})}\frac{\partial\Phi(\underline{u})}{\partial\underline{u}}\otimes M^{T}_{2k+1}(\underline{u})+\frac{\partial}{\partial\underline{u}}\otimes M^{T}_{2k+1}(\underline{u}) \tag{55}$$

$$=-\underline{u}\otimes M^{T}_{2k+1}(\underline{u})+\frac{\partial}{\partial\underline{u}}\otimes M^{T}_{2k+1}(\underline{u}).$$

Finally, letting $k = 0$ in equation (48) leads to the initial condition

$$M_{o}(\underline{u})=1. \tag{56}$$

Equations (51), (55), and (56) specify a recursive procedure for computing the set of matrices $\{M_{k}(\underline{u}): k=0,1,2,\ldots\}$. An examination of equations (48) and (49) reveals that the dimension of $M_{2k}(\underline{u})$ and $M_{2k+1}(\underline{u})$ are of dimension ($n^k$ x $n^k$) and ($n^{(k+!)}$ x $n^k$). Furthermore, since the vec (•) operator simply rearranges matrix elements in a prescribed fashion (see appendix A), it follows from equation set (50) that $\psi_{k}(\underline{u})$ can be easily evaluated once $M_k(\underline{u})$ is determined explicitly. At the $2k^{th}$ stage, actual computations involve ($n^k$ x $n^k$) dimensional matrices rather than ($n^{2k}$ x 1) dimensional vectors. This reduction in the exponential growth rate by a factor of 2 significantly reduces the complexity of required mathematical operations. For convenience, the pertinent equations and first few polynomial representations are summarized as follows:

$$M_o(\underline{u}) = 1, \tag{57a}$$

$$M_{2k+1}(\underline{u}) = -\underline{u} \otimes M_{2k}(\underline{u}) + \frac{\partial}{\partial \underline{u}} \otimes M_{2k}(\underline{u}), \tag{57b}$$

$$M_{2k+2}(\underline{u}) = -\underline{u} \otimes M^{T}_{2k+1}(\underline{u}) + \frac{\partial}{\partial \underline{u}} \otimes m^{T}_{2k+1}(\underline{u}), \tag{57c}$$

$$\underline{\psi}_k(\underline{u}) = vec\{M_k(\underline{u})\} \qquad\qquad k = 0,1,2,\ldots, \tag{57d}$$

$$M_o(\underline{u}) = 1, \tag{58a}$$

$$M_1(\underline{u}) = -\underline{u}, \tag{58b}$$

$$M_2(\underline{u}) = \underline{u}\underline{u}^T - I, \tag{58c}$$

$$M_3(\underline{u}) = C[I \otimes \underline{u}] - \underline{\psi}_2(\underline{u})\underline{u}^T, \tag{58d}$$

$$M_4(\underline{u}) = \underline{\psi}_2(\underline{u})\underline{\psi}^T_2(\underline{u}) - C[I \otimes M_2(\underline{u})]C - C. \tag{58e}$$

Here,

$$I = \sum_{i=1}^{n} \underline{e}_i(n)\underline{e}^T_i(n) \tag{59}$$

is the ($n$ x $n$) identity matrix. In addition,

$$C = I \otimes I + U(n,n), \tag{60}$$

where

$$U(n,n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{e_i(n)\underline{e}^T_j(n)\right\} \otimes \left\{\underline{e}_j(n)\underline{e}^T_i(n)\right\} \tag{61}$$

is the ($n^2$ x $n^2$) permutation matrix,[12] and

$$e_1(n) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \\ 0 \end{bmatrix}, \qquad e_2(n) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \\ 0 \end{bmatrix}, \qquad \ldots, \qquad \underline{e}_n(n) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \\ 0 \end{bmatrix} \tag{62}$$

are ($n$ x 1) unit vectors.

Returning to the problem at hand, a closed-form expression for the vector valued polynomial $\underline{\xi}_k(\underline{x};\hat{\underline{x}},\Gamma)$ follows immediately by combining equation set (38) with equation set (50). In particular,

$$\underline{\xi}_k(\underline{x};\hat{\underline{x}},\Gamma) = vec\left\{M_k\left(\Gamma^{-1/2}\{\underline{x}-\hat{\underline{x}}\}\right)\right\} \qquad k = 0,1,2,\ldots. \tag{63}$$

Substituting this expression into equation (43) then yields

$$\underline{\beta}_k = vec\left\{\int_{R_n} M_k\left(\Gamma^{-1/2}\{\underline{\tilde{x}}-\hat{\underline{x}}\}\right)p(\underline{\tilde{x}})d\underline{\tilde{x}}\right\} \qquad k = 0,1,2,\ldots. \tag{64}$$

Use of equations (40), (41), and equation set (58) allows explicit evaluation of $\underline{\beta}_k$ as shown below:

$$\underline{\beta}_o = 1, \tag{65a}$$

$$\underline{\beta}_1 = \underline{0} = (n \times 1)\textit{null vector}, \tag{65b}$$

$$\underline{\beta}_2 = \underline{0}\otimes\underline{0} = (n^2 \times 1)\textit{null vector}, \tag{65c}$$

$$\underline{\beta}_3 = -\left[\Gamma^{-1/2}\otimes\ldots\right]^3\int_{R_n}\left[(\underline{\tilde{x}}-\hat{\underline{x}})\otimes\ldots\right]^3 p(\underline{\tilde{x}})d\underline{\tilde{x}}, \tag{65d}$$

$$\underline{\beta}_4 = \left[\Gamma^{-1/2}\otimes\ldots\right]^4\left[\int_{R_n}\left[(\underline{\tilde{x}}-\hat{\underline{x}})\otimes\ldots\right]^4 p(\underline{\tilde{x}})d\underline{\tilde{x}} - vec\{C(\Gamma\otimes\Gamma)\} - vec(\Gamma)\otimes vec(\Gamma)\right]. \tag{65e}$$

Since each term in the multivariate Gram-Charlier series expansion of $p(\underline{x})$ takes the form of a scalar product of two vectors, the computation process can be simplified further by employing

various mathematical artifices. Details of the procedure, along with explicit evaluation of the first two non-trivial terms, are presented in appendix B.

## 2.3 AN INTEGRAL EQUATION SATISFIED BY THE MULTIVARIATE GRAM-CHARLIER EXPANSION FUNCTIONS

In section 2.1, the Gram-Charlier expansion functions were defined by the differential relationships expressed in equation set (37), or equivalently, in equation set (63). An alternative relationship in integral form may also be developed by using the well-known characteristic function formula for a $n$-dimensional normalized Gaussian PDF,[3] viz.,

$$\Phi(\underline{u}) = \frac{1}{(2\pi)^n} \int_{R_n} e^{j\underline{u}^T \underline{w} - 1/2(\underline{w}^T \underline{w})} d\underline{w}, \tag{66}$$

where $\Phi(\underline{u})$ is specified by equation (12). Differentiating this expression successively with respect to $\underline{u}$, and noting that

$$\frac{\partial}{\partial \underline{u}} \otimes e^{j\underline{u}^T \underline{w}} = j\underline{w} e^{j\underline{u}^T \underline{w}} \tag{67}$$

leads to the expression

$$\left[\left(\frac{\partial}{\partial \underline{u}}\right) \otimes \ldots\right]^k \Phi(\underline{u}) = \frac{j^k}{(2\pi)^n} \int_{R_n} [\underline{w} \otimes \ldots]^k e^{j\underline{u}^T \underline{w} - 1/2(\underline{w}^T \underline{w})} d\underline{w}. \tag{68}$$

An integral formula for $\underline{\psi}_R(\underline{u})$ is now obtained by dividing both sides of equation (68) with $\Phi(\underline{u})$, and then using equations (12) and (43). The result is

$$\underline{\psi}_k(\underline{u}) = \frac{j^k}{(2\pi)^{n/2}} \int_{R_n} [w \otimes \ldots]^k e^{1/2(\underline{u} + j\underline{w})^T (\underline{u} + j\underline{w})} d\underline{w}. \tag{69}$$

Combining equation set (38) with equation (69) also yields

$$\underline{\xi}_k(\underline{x}; \hat{\underline{x}}, \Gamma) = \frac{j^k}{(2\pi)^{n/2}} \int_{R_n} [w \otimes \ldots]^k e^{1/2\left\{\Gamma^{-1/2}(\underline{x} - \hat{\underline{x}}) + j\underline{w}\right\}^T \left\{\Gamma^{-1/2}(\underline{x} - \hat{\underline{x}}) + j\underline{w}\right\}} d\underline{w}. \tag{70}$$

To derive the integral equation for $\underline{\psi}_k(\underline{u})$, multiply both sides of equation (69) by $N\left(\underline{u}; \hat{\underline{u}}, [I + \Omega]^{-1}\right)$ and integrate the resulting expression over $R_n$ to obtain

$$\int_{R_n} N\left(\underline{u};\underline{\hat{u}},[I+\Omega]^{-1}\right)\underline{\psi}_k(\underline{u})d\underline{u}$$

$$= \frac{j^k}{(2\pi)^{n/2}}\int_{R_n}\int_{R_n}\frac{\sqrt{|I+\Omega|}}{(2\pi)^{n/2}}[w\otimes\ldots]^k\; e^{-1/2\left\{(\underline{u}-\underline{\hat{u}})^T[I+\Omega](\underline{u}-\underline{\hat{u}})-(\underline{u}+j\underline{w})^T(\underline{u}+j\underline{w})\right\}}d\underline{w}d\underline{u}. \tag{71}$$

Here, it is assumed that $\Omega$ is a positive definite symmetric matrix. This assumption insures that the integrals appearing on the right-hand side of equation (71) are both absolutely convergent, thus permitting the order of integration to be interchanged. Performing this operation allows equation (70) to be rewritten in the form

$$\int_{R_n} N\left(\underline{u};\underline{\hat{u}},[I+\Omega]^{-1}\right)\underline{\psi}_k(\underline{u})d\underline{u}$$

$$= \frac{j^k}{(2\pi)^{n/2}}\int_{R_n}\sqrt{\frac{|I+\Omega|}{|\Omega|}}\,[\underline{w}\otimes\ldots]^k e^{1/2(\underline{\hat{u}}+j\underline{w})^T[I+\Omega^{-1}](\underline{\hat{u}}+j\underline{w})}\left[\frac{\sqrt{|\Omega|}}{(2\pi)^{n/2}}\int_{R_n} e^{-1/2\left(\underline{u}-\underline{u}^*\right)^T\Omega\left(\underline{u}-\underline{u}^*\right)}d\underline{u}\right]d\underline{w}\,, \tag{72}$$

where

$$\underline{u}^* = \underline{\hat{u}}+\Omega^{-1}(\underline{\hat{u}}+j\underline{w}). \tag{73}$$

Since the integral enclosed by the square brackets on the right-hand side of equation (72) is equal to unity, it follows that

$$\int_{R_n} N\left(\underline{u};\underline{\hat{u}},[I+\Omega]^{-1}\right)\underline{\psi}_k(\underline{u})d\underline{u} = \frac{j^k}{(2\pi)^{n/2}}\int_{R_n}\sqrt{\frac{|I+\Omega|}{|\Omega|}}\,[\underline{w}\otimes\ldots]^k\; e^{1/2(\underline{\hat{u}}+j\underline{w})^T[I+\Omega^{-1}](\underline{\hat{u}}+j\underline{w})}d\underline{w}. \tag{74}$$

Under the assumption that $\Omega$ is positive definite and symmetric, the matrix $\left[I+\Omega^{-1}\right]$ will also possess these properties; consequently, it has a square root decomposition of the form

$$\left[I+\Omega^{-1}\right]=\left[I+\Omega^{-1}\right]^{1/2}\left[I+\Omega^{-1}\right]^{1/2}, \tag{75a}$$

$$\left\{\left[I+\Omega^{-1}\right]^{1/2}\right\}^T=\left[I+\Omega^{-1}\right]^{1/2}. \tag{75b}$$

A new variable of integration can then be introduced into equation (74) via the formula $\underline{z}=\left[I+\Omega^{-1}\right]^{1/2}\underline{w}$. Performing the necessary algerbraic manipulations and using the "Mixed Multiplication Rule" described in appendix A {see equation (A-8)} yields

$$\int_{R_n} N\left(\underline{u};\underline{\hat{u}},[I+\Omega]^{-1}\right)\underline{\psi}_k(\underline{u})d\underline{u}$$

$$\left[\left[I+\Omega^{-1}\right]^{-1/2}\otimes\ldots\right]^k \frac{j^k}{(2\pi)^{n/2}}\int_{R_n}[\underline{z}\otimes\ldots]^k\, e^{1/2\left(\left[I+\Omega^{-1}\right]^{1/2}\underline{\hat{u}}+j\underline{z}\right)^T\left(\left[I+\Omega^{-1}\right]^{1/2}\underline{\hat{u}}+j\underline{z}\right)}d\underline{z}\,. \tag{76}$$

The desired integral equation for $\underline{\psi}_k(\underline{u})$ follows immediately by comparing the right-hand side of equation (76) with equation (69); viz.,

$$\int_{R_n} N\left(\underline{u};\underline{\hat{u}},[I+\Omega]^{-1}\right)\underline{\psi}_k(\underline{u})d\underline{u}$$

$$=\left[\left[I+\Omega^{-1}\right]^{-1/2}\otimes\ldots\right]^k \underline{\psi}_k\left(\left[I+\Omega^{-1}\right]^{1/2}\underline{\hat{u}}\right) \qquad k=0,1,2,\ldots. \tag{77}$$

The result embodied in equation set (77) will be used in section 3 of this report to provide a closed-form solution to the problem of detecting random multivariate signals embedded in Gaussian noise.

## 2.4 DISCUSSION OF SECTION 2 RESULTS

With the aid of the $n$-dimensional Dirac delta function representation given by equation (36), it was a straightforward task to obtain the Gram-Charlier series expansion for an arbitrary multivariate PDF as depicted in equation (42). Similarities between the univariate PDF expansion and its multivariate counterpart are clearly evident. In the univariate case, the expansion coefficients are all scalar quantities; the first three values being 1,0, and 0, respectively, while the remaining coefficients depend successively upon progressively higher-order moments starting with third order.[3] An examination of equation set (65) reveals that the same pattern prevails in the multivariate case, only now the expansion coefficients are vectors of exponentially increasing dimension. Furthermore, using only the first term of the Gram-Charlier series expansion to approximate an actual PDF, whether in the univariate or multivariate case, will result in the equality of corresponding moments up to and including second order. As might be expected, including successively more terms of the series expansion will result in the equality of progressively higher order moments.

# 3. MULTIVARIATE RANDOM SIGNAL DETECTION IN GAUSSIAN NOISE

## 3.1 FORMULATION OF DETECTION PROBLEM

The sonar detection problem is usually formulated as a binary statistical hypothesis test.[13,18] Under the null hypothesis $H_o$, observations (i.e., data describing the output of acoustic sensors) consist of noise alone; under the alternative hypothesis $H_1$, observations consist of noise and signals that represent responses of the acoustic sensors to the presence of a target or other interesting object. In practice, the signals and noise are usually time-varying functions; however, the time parameter can actually be suppressed without loss of generality via judicious application of the Karhunen-Loeve expansion.[13] Consequently, only the single-time epoch problem will be considered; the results being readily extended to time-varying situations without undue complication. With this in mind, the aforementioned detection problem is mathematically described as follows:

$$H_o : \underline{r} = \underline{n}, \tag{78a}$$

$$H_1 : \underline{r} = \underline{s} + \underline{n}, \tag{78b}$$

where

$$\underline{r} = \left[r_1, r_2, \ldots, r_n\right]^T, \tag{79}$$

$$\underline{s} = \left[s_1, s_2, \ldots, s_n\right]^T, \tag{80}$$

$$\underline{n} = \left[n_1, n_2, \ldots, n_n\right]^T. \tag{81}$$

Here $r_i$ denotes the output of the $i^{\text{th}}$ sensor element (e.g., the response of the $i^{\text{th}}$ transducer of an acoustic array), and $s_i$ and $n_i$ are the corresponding signal and noise components, respectively. The vector $\underline{r}$ is a random process that describes the output (voltage) of an array of acoustic transducers comprising the data. The signal and noise vectors are assumed to be uncorrelated random processes, with the noise having a multivariate Gaussian PDF given by

$$P_{\underline{n}}(\underline{n}) = N\left(\underline{n}; \hat{\underline{n}}, \Gamma_{\underline{n}}\right). \tag{82}$$

The signal PDF, denoted by $p_{\underline{s}}(\underline{s})$, is an arbitrary multivariate distribution that satisfies the relations

$$\hat{\underline{s}} = \int_{R_n} \underline{s} p_{\underline{s}}(\underline{s}) d\underline{s} \tag{83a}$$

$$\Gamma_{\underline{s}} = \int_{R_n} (\underline{s} - \hat{\underline{s}})(\underline{s} - \hat{\underline{s}})^T p_{\underline{s}}(\underline{s}) d\underline{s} \tag{83b}$$

and, additionally, fulfills all the usual requirements[3] of a legitimate PDF.

The optimal solution to the binary detection problem specified by equation (78) is well-known and readily available in the literature.[2,4,10,13] It is called the "likelihood ratio" test, denoted by $\Lambda(\underline{r})$, and can be expressed as

$$\Lambda(\underline{r}) = \int_{R_n} \frac{p_{\underline{n}}(\underline{r} - \underline{s})}{p_{\underline{n}}(\underline{r})} p_{\underline{s}}(\underline{s}) d\underline{s}. \tag{84}$$

When both signal and noise vectors are Gaussian random processes, the integral appearing on the right-hand side of equation (84) can be evaluated explicitly. In this case, a simple canonical representation for the likelihood ratio is obtained; viz.,

$$\Lambda_g(\underline{r}) = \frac{N\left(\underline{r}; \{\hat{\underline{s}} + \hat{\underline{n}}\}, \left[\Gamma_{\underline{s}} + \Gamma_{\underline{n}}\right]\right)}{N\left(\underline{r}; \hat{\underline{n}}, \Gamma_{\underline{n}}\right)}. \tag{85}$$

The subscript "$g$" is used here to underscore the Gaussian character of both signal and noise. The case involving non-Gaussian signals embedded in Gaussian noise yields a more formidable expression for the likelihood ratio that is not amenable to closed-form evaluation. As a result, some type of approximation technique is required. One particularly attractive procedure is described in the next section.

## 3.2 DERIVATION OF A CANONICAL REPRESENTATION FOR THE LIKELIHOOD RATIO

The derivation of a canonical representation for the likelihood ratio $\Lambda(\underline{r})$ is based on the result embodied in equation (84). In that expression, closed-form evaluation of the integral is prohibited because $p_{\underline{s}}(\underline{s})$ is specified as an arbitrary PDF. However, this difficulty is circumvented by expanding $p_{\underline{s}}(\underline{s})$ in a multivariate Gram-Charlier series as described by equation (42); viz.,

$$p_{\underline{s}}(\underline{s}) = N\left(\underline{s}; \hat{\underline{s}}, \Gamma_{\underline{s}}\right) \sum_{k=0}^{\infty} \frac{1}{k!} \underline{\beta}_k^T \underline{\xi}_k\left(\underline{s}; \hat{\underline{s}}, \Gamma_{\underline{s}}\right), \tag{86}$$

where

$$\underline{\beta}_k = \int_{R_n} \underline{\xi}_k\left(\underline{s}; \hat{\underline{s}}, \Gamma_{\underline{s}}\right) p_{\underline{s}}(\underline{s}) d\underline{s}, \tag{87}$$

and $\underline{\xi}_k(\underline{s};\hat{\underline{s}},\Gamma_{\underline{s}})$ are the ($n^k$ x 1 ) vector valued functions defined by equation set (37). If equations (82) and (86) are substituted into equation (84), and the summation and integration processes interchanged (which is permissible since both are assumed to be absolutely convergent), it follows that

$$\Lambda(r) = \sum_{k=0}^{\infty} \frac{\underline{\beta}_k^T}{k\,!} \int_{R_n} \frac{N(\{\underline{r}-\underline{s}\};\hat{\underline{n}},\Gamma_{\underline{n}})N(\underline{s};\hat{\underline{s}},\Gamma_{\underline{s}})}{N(\underline{r};\hat{\underline{n}},\Gamma_{\underline{n}})} \underline{\xi}_k(\underline{s};\hat{\underline{s}},\Gamma_{\underline{s}})d\underline{s}. \tag{88}$$

Next, observe that the identity

$$\frac{N(\{\underline{r}-\underline{s}\};\hat{\underline{n}},\Gamma_{\underline{n}})N(\underline{s};\hat{\underline{s}},\Gamma_{\underline{s}})}{N(\underline{r};\hat{\underline{n}},\Gamma_{\underline{n}})} = \frac{\Lambda_g(\underline{r})}{\sqrt{|\Gamma_{\underline{s}}|}} N\left(\Gamma_{\underline{s}}^{-1/2}\{\underline{s}-\hat{\underline{s}}\};\hat{\underline{u}},[I+\Omega]^{-1}\right), \tag{89}$$

where

$$\hat{\underline{u}} = \left[I+\Omega^{-1}\right]^{-1}\Gamma_{\underline{s}}^{-1/2}(\underline{r}-\hat{\underline{s}}-\hat{\underline{n}}), \tag{90}$$

and

$$\Omega = \Gamma_{\underline{s}}^{1/2}\Gamma_{\underline{n}}^{-1}\Gamma_{\underline{s}}^{1/2}, \tag{91}$$

can be readily deduced via straightforward algebraic manipulation and use of equation (85). If equations (38) and (89) are now substituted into the integral appearing on the right-hand side of equation (88), and the variable of integration changed via the formula $\underline{u} = \Gamma_{\underline{s}}^{-1/2}(\underline{s}-\hat{\underline{s}})$,

$$\Lambda(\underline{r}) = \Lambda_g(\underline{r}) \sum_{k=0}^{\infty} \frac{\underline{\beta}_k^T}{k\,!} \int_{R_n} N\left(\underline{u};\hat{\underline{u}},[I+\Omega]^{-1}\right)\underline{\psi}_k(\underline{u})d\underline{u}. \tag{92}$$

The integral appearing in equation (92) is identical to the one shown in equation set (77), and the matrix $\Omega$ defined by equation (91) is symmetric and positive definite. As a result, equation (92) may be rewritten in the form

$$\Lambda(\underline{r}) = \Lambda_g(\underline{r}) \sum_{k=0}^{\infty} \frac{\underline{\beta}_k^T}{k\,!} \left[\left[I+\Omega^{-1}\right]^{-1/2} \otimes \ldots\right]^k \underline{\psi}_k\left(\left[I+\Omega^{-1}\right]^{1/2}\hat{\underline{u}}\right). \tag{93}$$

Alternatively,

$$\Lambda(\underline{r}) = \Lambda_g(\underline{r})\sum_{k=0}^{\infty}\frac{\underline{\beta}_k^T}{k!}\left[\left(\Gamma_{\underline{s}}^{1/2}\left[\Gamma_{\underline{s}}+\Gamma_{\underline{n}}\right]^{-1/2}\right)\otimes\ldots\right]^k \underline{\xi}_k\left(\underline{r};\{\hat{\underline{s}}+\hat{\underline{n}}\},\left[\Gamma_{\underline{s}}+\Gamma_{\underline{n}}\right]\right), \tag{94}$$

where the latter expression follows by substituting equations (90) and (91) into equation (74), algebraically manipulating the result, and then using equation (70) along with the "Mixed Multiplication Rule" described in appendix A. This is the desired canonical representation for the likelihood ratio. It might be noted that the first two terms of the series given by equation (94) are explicitly evaluated in appendix B (see equations (B-20) through (B-25)).

### 3.3 DISCUSSION OF SECTION 3 RESULTS

It is noteworthy that the infinite series appearing in equation (94) can be summed explicitly in the absence of the matrix Kronecker product $\left[\left(\Gamma_{\underline{s}}^{1/2}\left[\Gamma_{\underline{s}}+\Gamma_{\underline{n}}\right]^{-1/2}\right)\otimes\ldots\right]^k$. In particular, substituting equation sets (33) and (38) into equation (86) reveals that

$$\sum_{k=0}^{\infty}\frac{\underline{\beta}_k^T}{k!}\underline{\xi}_k\left(\underline{r};\{\hat{\underline{s}}+\hat{\underline{n}}\},\left[\Gamma_{\underline{s}}+\Gamma_{\underline{n}}\right]\right) = \sqrt{\frac{\left|\Gamma_{\underline{s}}\right|}{\left|\Gamma_{\underline{s}}+\Gamma_{\underline{n}}\right|}}\,\frac{p_{\underline{s}}\left(\hat{\underline{s}}+\Gamma_{\underline{s}}^{1/2}\left[\Gamma_{\underline{s}}+\Gamma_{\underline{n}}\right]^{-1/2}\{\underline{r}-\hat{\underline{s}}-\hat{\underline{n}}\}\right)}{N\left(\underline{r};\{\hat{\underline{s}}+\hat{\underline{n}}\},\left[\Gamma_{\underline{s}}+\Gamma_{\underline{n}}\right]\right)}. \tag{95}$$

This result suggests some interesting possibilities for simply approximating $\Lambda(\underline{r})$ in situations where the signal-to-noise ratio is high. For example, an intuitively pleasing formula is obtained by introducing the approximation $\Gamma_{\underline{s}}+\Gamma_{\underline{n}}\approx\Gamma_{\underline{s}}$. Under this assumption, equations (82), (85), (94), and (95) may be combined and subsequently manipulated to produce the expression

$$\Lambda(\underline{r}) \approx \frac{p_{\underline{s}}(\underline{r}-\hat{\underline{n}})}{p_{\underline{n}}(\underline{r})}. \tag{96}$$

Comparing equation (84) with equation (96) reveals that the noise PDF takes on Dirac delta function characteristics as the signal strength increases to the point of overwhelming the noise. This is expected, since the condition $\Gamma_{\underline{s}}+\Gamma_{\underline{n}}\rightarrow\Gamma_{\underline{s}}$ has a mathematically analogous effect on the integration process depicted in equation (84) as the condition $\Gamma_{\underline{n}}\rightarrow[0]$. Recalling equation (82) and the fact that

$$\lim_{\Gamma_{\underline{n}}\rightarrow[0]} N\left(\{\underline{r}-\underline{s}\};\hat{\underline{n}},\Gamma_{\underline{n}}\right)\rightarrow\delta(\underline{r}-\underline{s}-\hat{\underline{n}}), \tag{97}$$

readily explains the approximation given by equation (96).

Equation (95) may also be used to confirm earlier results depicted by equation (85). Indeed, for Gaussian signal and noise vectors, the right-hand side of equation (95) reduces to unity, while all terms on the left-hand side vanish except for the one corresponding to $k = 0$. In this case, the equivalence between (85) and (94) is manifest. Finally, even if no constraints are imposed on the signal-to-noise ratio, the canonical representation depicted in equation (94) may be recast in the form

$$\Lambda(\underline{r}) = \sqrt{\frac{\left|\Gamma_{\underline{s}}\right|}{\left|\Gamma_{\underline{s}} + \Gamma_{\underline{n}}\right|}}\ \frac{p_{\underline{s}}\left(\underline{\hat{s}} + \Gamma_{\underline{s}}^{1/2}\left[\Gamma_{\underline{s}} + \Gamma_{\underline{n}}\right]^{-1/2}\left\{\underline{r} - \underline{\hat{s}} - \underline{\hat{n}}\right\}\right)}{p_{\underline{n}}(\underline{r})}$$
$$- \Lambda_g(\underline{r}) \sum_{k=0}^{\infty} \frac{\underline{\beta}_k^T}{k!}\left[\Gamma_{\underline{s}}^{1/2} \otimes \ldots\right]^k Q_k \underline{\xi}_k\left(\underline{r}; \left\{\underline{\hat{s}} + \underline{\hat{n}}\right\}, \left[\Gamma_{\underline{s}} + \Gamma_{\underline{n}}\right]\right), \tag{98}$$

where

$$Q_k = \left[\Gamma_{\underline{s}}^{-1/2} \otimes \ldots\right]^k - \left[\left[\Gamma_{\underline{s}} + \Gamma_{\underline{n}}\right]^{-1/2} \otimes \ldots\right]^k. \tag{99}$$

In practical applications, situations may arise wherein an $N$-term series approximation based on equation (98) will provide more accuracy than a similar type approximation based on equation (94). Of course, the opposite situation may also arise under appropriate circumstances. Ostensibly, equation (94) should prove to be more utilitarian under low signal-to-noise ratio conditions, while equation (98) is preferred when high signal-to-noise ratio conditions prevail. Having both representations available thus provides a greater degree of flexibility for dealing with specific problems.

## 4. SUMMARY AND CONCLUSIONS

Section 2 of this report extended the well-known Gram-Charlier series expansion technique for an arbitrary univariate PDF to the multivariate case. The approach employed here deliberately avoided explicit use of tensor analysis and multivariate Hermite polynomials, both of which are cumbersome to apply, even under the best of circumstances. The desired expansion was obtained by developing a suitable infinite series representation for the $n$-dimensional Dirac delta function, and then judiciously using Kronecker product formulae and matrix calculus rules to manipulate the result.

The analogy between a univariate PDF series expansion and its multivariate counterpart is clearly evident. Expansion coefficients in the univariate case are all scalar quantities, the first three values being 1,0, and 0, respectively. The remaining coefficients depend, in succession, on progressively higher order moments starting with the third order. For the multivariate case, a similar pattern exists; however, the expansion coefficients are now vector quantities whose

dimension increases in an exponential manner with the summation index. In both cases, only a single term in the series expansion is needed to exactly replicate any Gaussian PDF. Under such circumstances, all the expansion coefficients vanish except the first. When the actual PDF is non-Gaussian, using a one-term series approximation leads to the equality of corresponding moments up to and including second order. As might be expected, retaining successively more terms in the series approximation of the actual PDF will result in the equality of progressively higher order moments.

While the computational procedure for explicitly evaluating the multivariate expansion coefficients is by no means trivial, it is nevertheless tractable. A particularly attractive feature of the formulae is that the statistical mean and covariance associated with the actual PDF always appear in their compact vector and matrix formats, respectively. The need to explicitly depict individual vector components or matrix elements has been avoided by using Kronecker products. This artifice greatly simplifies expressions that would otherwise be unwieldy and not at all amenable to analysis and/or interpretation.

An important application of the multivariate Gram-Charlier series expansion technique is in solving the binary detection problem for multivariate signals embedded in Gaussian background noise. Section 3 of this report addressed that problem in the context of sonar array signal processing. The solution was specified as a binary statistical hypothesis test; viz., the "likelihood ratio test." A canonical representation for the likelihood ratio was developed in the form of an infinite series whose individual terms depend on the received measurement vector together with the signal and noise statistics. A closed-form expression for each term was subsequently obtained via use of an integral equation satisfied by the multivariate Gram-Charlier expression coefficients. This description reveals manifest similarities between the multivariate and univariate canonical representations. In fact, it is easily demonstrated that the former reduces to the latter as the dimensionality parameter $n$ approaches unity.

Another important consequence of evaluating the individual terms of the infinite series in closed-form is that it facilitates the derivation of likelihood ratio approximations that are suitable for use under various operating conditions. In particular, the aforementioned description of individual terms allows the original infinite series to be decomposed into two other infinite series, one of which is summable to a recognizable function. As a result, the likelihood ratio can be represented in either of two canonical forms. The forms are equivalent and of identical composition; viz., both are expressed as infinite series. Ostensibly, the convergence characteristics of these two series are quite different. When the signal-to-noise ratio is high, one series appears to converge rapidly, while the other does not. When the signal-to-noise ratio is low, this pattern seems to reverse itself. Consequently, in practical applications, judiciously choosing the appropriate canonical representation before making an $N$-term series approximation should yield a more accurate description of the actual likelihood ratio. Having both representations available obviously enhances the capability to address specific problems more effectively.

# APPENDIX A

## KRONECKER PRODUCT FORMULAE AND MATRIX CALCULUS RULES

Presented in this appendix are some formulae for manipulating Kronecker products and for performing matrix calculus. The results are presented without proof. Interested readers are referred to Graham [12] for detailed derivations and a comprehensive treatment of the subject material.

## DEFINITION OF THE KRONECKER PRODUCT

Consider a matrix $A = [a_{ij}]$ of order ($m$ x $p$) and a matrix $B = [b_{ij}]$ of order ($r$ x $s$). The Kronecker product of the two matrices, denoted by $A \otimes B$, is defined as the partitioned matrix:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1p}B \\ a_{21}B & a_{22}B & \dots & a_{2p}B \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{m1}B & a_{m2}B & \dots & a_{mp}B \end{bmatrix}. \qquad \text{(A-1)}$$

$A \otimes B$ is seen to be a matrix of order ($mr$ x $ps$). It has $mp$ blocks; the $(i,j)^{th}$ block is the matrix $a_{ij}B$ of order ($r$ x $s$).

Some properties and rules for Kronecker products are:

1. $(\alpha A) \otimes (\beta B) = \alpha\beta(A \otimes B)$ for any scalars $\alpha, \beta$. (A-2)

2. If $A$ and $B$ are both ($m$ x $p$), and $C$ is ($r$ x $s$), then
$(A + B) \otimes C = A \otimes C + B \otimes C$. (A-3)

3. If $A$ is ($m$ x $p$) and both $B$ and $C$ are ($r$ x $s$), then
$A \otimes (B + C) = A \otimes B + A \otimes C$. (A-4)

4. $A \otimes (B \otimes C) = (A \otimes B) \otimes C$. (A-5)

5. $(A \otimes B)^T = A^T \otimes B^T$. (A-6)

6. If $A$ and $B$ are both ($m$ x $m$), then
$tr(A \otimes B) = tr(A)\, tr(B)$, (A-7)

where $tr(\ )$ is the trace operator.

7. If $A$ is $(m \times p)$, $B$ is $(q \times r)$, $C$ is $(p \times s)$, and $D$ is $(r \times v)$, then
$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$. (A-8)

8. If $A$ and $B$ are both nonsingular, then
$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$. (A-9)

9. If $A$ is $(m \times p)$ and $B$ is $(q \times r)$, then
$A \otimes B = U(m,q)\,(B \otimes A)\,U(r,p)$, (A-10)

where

$$U(k,l) = \sum_{i=1}^{k} \sum_{j=1}^{l} \left\{\underline{e}_i(\underline{k})\underline{e}_j^T(l)\right\} \otimes \left\{\underline{e}_j(l)\underline{e}_i^T(k)\right\} \tag{A-11}$$

is the $(kl \times kl)$ permutation matrix[12] defined for all integer values of $k$ and $l$, and

$$e_1(n) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \\ 0 \end{bmatrix}, \qquad e_2(n) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \\ 0 \end{bmatrix}, \qquad \ldots, \qquad \underline{e}_n(n) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \\ 1 \end{bmatrix} \tag{A-12}$$

are $(n \times 1)$ unit vectors defined for all integer values of n.

Let

$$\frac{\partial}{\partial \underline{u}} = \left[\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \ldots, \frac{\partial}{\partial u_n}\right]^T \qquad \textit{for } u = (\tilde{x}, x), \tag{A-13}$$

and assume that $\Gamma$ is a symmetric, positive definite $(n \times n)$ matrix. Under these assumptions, it is easily shown[19] that $\Gamma$ has the square root decomposition

$$\Gamma = \Gamma^{1/2}\Gamma^{1/2}, \tag{A-14}$$

$$\left(\Gamma^{1/2}\right)^T = \Gamma^{1/2}. \tag{A-15}$$

Now observe that

$$\left(\frac{\partial}{\partial \underline{x}}\right)^T \Gamma \left(\frac{\partial}{\partial \underline{x}}\right) = \left(\Gamma^{1/2}\frac{\partial}{\partial \underline{x}}\right)^T \left(\Gamma^{1/2}\frac{\partial}{\partial \underline{x}}\right). \tag{A-16}$$

Consequently, the square of this scalar differential operator can be expressed in the form

$$\left[\left(\frac{\partial}{\partial \underline{x}}\right)^T \Gamma \left(\frac{\partial}{\partial \underline{x}}\right)\right]^2 = \left[\left(\Gamma^{1/2}\frac{\partial}{\partial \underline{x}}\right)^T \left(\Gamma^{1/2}\frac{\partial}{\partial \underline{x}}\right)\right] \otimes \left[\left(\Gamma^{1/2}\frac{\partial}{\partial \underline{x}}\right)^T \left(\Gamma^{1/2}\frac{\partial}{\partial \underline{x}}\right)\right], \tag{A-17}$$

since the Kronecker product of two scalar quantities is identical to the ordinary scalar product. Using equation (A-8) allows equation (A-17) to be rewritten as

$$\left[\left(\frac{\partial}{\partial \underline{x}}\right)^T \Gamma \left(\frac{\partial}{\partial \underline{x}}\right)\right]^2 = \left[\left(\Gamma^{1/2}\frac{\partial}{\partial \underline{x}}\right)^T \otimes \ldots\right]^2 \left[\left(\Gamma^{1/2}\frac{\partial}{\partial \underline{x}}\right) \otimes \ldots\right]^2, \tag{A-18}$$

where the notation

$$[A \otimes \ldots]^k = \underbrace{A \otimes A \otimes \ldots \otimes A}_{k\ times} \qquad k = 0,1,2,\ldots, \tag{A-19}$$

has been employed. Repeating this process yields the following result by induction:

$$\left[\left(\frac{\partial}{\partial \underline{x}}\right)^T \Gamma \left(\frac{\partial}{\partial \underline{x}}\right)\right]^k = \left[\left(\Gamma^{1/2}\frac{\partial}{\partial \underline{x}}\right)^T \otimes \ldots\right]^k \left[\left(\Gamma^{1/2}\frac{\partial}{\partial \underline{x}}\right) \otimes \ldots\right]^k. \tag{A-20}$$

## *vec* ( ) OPERATOR

If

$$A = \left[\underline{a}_1 \vdots \underline{a}_2 \vdots \ldots \vdots \underline{a}_p\right] \tag{A-21}$$

is an ($m$ x $p$) matrix partitioned as shown, then

$$vec(A) = \begin{bmatrix} \underline{a}_1 \\ --- \\ \underline{a}_2 \\ --- \\ \cdot \\ \cdot \\ \cdot \\ \\ --- \\ \underline{a}_p \end{bmatrix} \qquad \text{(A-22)}$$

is a ($mp$ x 1) vector. Some useful formulae connecting the *vec* of a matrix product with the Kronecker product and the trace of a matrix product are specified below:

1. If $B$ is ($m$ x $n$), $C$ is ($n$ x $q$), and $D$ is ($q$ x $s$), then
   $vec(BCD) = (D^T \otimes B)\ vec(C).$ (A-23)

2. If $B$ is ($m$ x $n$) and $\underline{u}$ is a ($n$ x 1) vector, then
   $vec(B \otimes \underline{u}) = vec(B) \otimes \underline{u}$ (A-24)
   $vec(\underline{u}\underline{u}^T) = \underline{u} \otimes \underline{u}.$ (A-25)

3. If $B$ is ($m$ x $n$) and $C$ is ($n$ x $m$), then
   $\{vec(B^T)\}^T\{vec(C)\} = tr(BC).$ (A-26)

**DIFFERENTIATION FORMULAE FOR KRONECKER PRODUCTS**

1. If $A$ is ($m$ x $p$), $B$ is ($p$ x $q$), and $\underline{u}$ is a ($n$ x 1) vector, then

$$\frac{\partial}{\partial \underline{u}} \otimes \{AB\} = \left(\frac{\partial}{\partial \underline{u}} \otimes A\right)B + \{I \otimes A\}\left(\frac{\partial}{\partial u} \otimes \underline{B}\right). \qquad \text{(A-27)}$$

2. If $A$ is ($m$ x $p$), $B$ is ($q$ x $r$), and $\underline{u}$ is a ($n$ x 1) vector, then

$$\frac{\partial}{\partial \underline{u}} \otimes \{A \otimes B\} = \left(\frac{\partial}{\partial \underline{u}} \otimes A\right) \otimes B + \{I \otimes U(m,q)\}\left\{\left(\frac{\partial}{\partial \underline{u}} \otimes B\right) \otimes A\right\}U(r,p). \qquad \text{(A-28)}$$

In equations (A-27) and (A-28), $I$ represents the ($n$ x $n$) identity matrix, while $U(m,q)$ and $U(r,p)$ represent permutation matrices defined by equation (A-11). Some special results are given as follows:

If $A$ is a constant ($m$ x $n$) matrix and $\underline{u}$ is a ($n$ x 1) vector, then

$$vec(\underline{u}) = \underline{u}\,, \tag{A-29}$$

$$vec(\underline{u}^T) = \underline{u}\,, \tag{A-30}$$

$$vec(\underline{u}^T A^T) = A\underline{u}\,, \tag{A-31}$$

$$\frac{\partial}{\partial \underline{u}} \otimes \underline{u}^T = I, \tag{A-32}$$

$$\frac{\partial}{\partial \underline{u}} \otimes \underline{u} = vec(I), \tag{A-33}$$

$$\frac{\partial}{\partial \underline{u}} \otimes \left\{\underline{u}^T \underline{u}\right\} = 2\underline{u}, \tag{A-34}$$

$$\frac{\partial}{\partial \underline{u}} \otimes \left\{A\underline{u}\right\} = vec(A), \tag{A-35}$$

$$\frac{\partial}{\partial \underline{u}} \otimes \left\{\underline{u}^T A^T\right\} = A^T. \tag{A-36}$$

If $A$ is a constant ($n$ x $n$) matrix and $\underline{u}$ is a ($n$ x 1) vector, then

$$\frac{\partial}{\partial \underline{u}} \otimes \left\{\underline{u}^T A\underline{u}\right\} = A\underline{u} + A^T \underline{u}. \tag{A-37}$$

## APPENDIX B
## COMPUTATION OF THE FIRST TWO NON-TRIVIAL TERMS OF THE MULTIVARIATE GRAM-CHARLIER SERIES

In this appendix, the first two non-trivial terms appearing in the Gram-Charlier series given by equation (42) are evaluated. Since each term in that series takes the form of a scalar product of two vectors, various mathematical artifices may be employed to simplify the calculations. To this end, observe that the first two non-trivial terms are associated with the indices $k = 3$ and $k = 4$. Those corresponding to $k = 0$, $k = 1$, and $k = 2$ are trivial to compute; their numerical values being 1, 0, and 0, respectively. Focusing on the non-trivial term with index $k = 3$, the evaluation process begins by using the formula

$$\underline{\psi}_2(\underline{u}) = \underline{u} \otimes \underline{u} - vec(I), \tag{B-1}$$

which follows from equations (57d), (58c), and (A-25). Substituting equation (B-1) into equation (58d), and combining that result with equations (57d), (A-23), and (A-24) yields

$$\underline{\psi}_3(\underline{u}) = \{I \otimes C\}\{vec(I) \otimes \underline{u}\} + \underline{u} \otimes vec(I) - \underline{u} \otimes \underline{u} \otimes \underline{u}. \tag{B-2}$$

Aided by equations (B-2) and (A-8), along with the relations

$$\underline{\beta}_3^T \{I \otimes C\}\{vec(I) \otimes \underline{u}\} = \underline{\beta}_3^T \{2vec(I) \otimes \underline{u}\}, \tag{B-3}$$

and

$$\underline{\beta}_3^T \{\underline{u} \otimes vec(I)\} = \underline{\beta}_3^T \{vec(I) \otimes \underline{u}\}, \tag{B-4}$$

it can be shown that

$$\underline{\beta}_3^T \underline{\psi}_3(\Gamma^{1/2}\{\underline{x} - \underline{\hat{x}}\}) = \left\{\int_{R_n} \left[(\underline{\tilde{x}} - \underline{\hat{x}}) \otimes \ldots\right]^3 p(\underline{\tilde{x}}) d\underline{\tilde{x}}\right\}^T \left\{\left[\underline{z} \otimes \underline{z} - 3vec(\Gamma^{-1})\right] \otimes \underline{z}\right\}, \tag{B-5}$$

where

$$\underline{z} = \Gamma^{-1}\{\underline{x} - \underline{\hat{x}}\}. \tag{B-6}$$

From equations (A-24) and (A-25) it also follows that

$$\left[\underline{z} \otimes \underline{z} - 3vec(\Gamma^{-1})\right] \otimes \underline{z} = vec\left\{\left[\underline{z}\underline{z}^T - 3\Gamma^{-1}\right] \otimes \underline{z}\right\}, \tag{B-7}$$

and

$$\int_{R_n}\left[(\underline{\tilde{x}}-\underline{\hat{x}})\otimes\ldots\right]^3 p(\underline{\tilde{x}})d\underline{\tilde{x}} = vec\left(\Delta_3^T\right), \tag{B-8}$$

where

$$\Delta_3 = \int_{R_n}\left[(\underline{\tilde{x}}-\underline{\hat{x}})(\underline{\tilde{x}}-\underline{\hat{x}})^T\right]\otimes(\underline{\tilde{x}}-\underline{\hat{x}})^T p(\underline{\tilde{x}})d\underline{\tilde{x}} \tag{B-9}$$

is the matrix of third order central moments associated with $p(\underline{x})$. Finally, substituting equations (B-7) and (B-8) into equation (B-5), and using equation set (38) and equation (A-26) leads to the desired result; viz.,

$$\underline{\beta}_3^T\underline{\xi}_3(\underline{x};\underline{\hat{x}},\Gamma) = tr\left\{\Delta_3\left[\left(zz^T-3\Gamma^{-1}\right)\otimes z\right]\right\}. \tag{B-10}$$

Computation of the non-trivial term associated with index $k = 4$ is initiated by using the formula

$$\underline{\psi}_2(\underline{u}) = \frac{C}{2}\underline{\psi}_2(\underline{u}), \tag{B-11}$$

in conjunction with equation (58e) to obtain

$$M_4(\underline{u}) = \frac{C}{2}\left[\underline{\psi}_2(\underline{u})\underline{\psi}_2^T(\underline{u}) - 4I\otimes M_2(\underline{u}) - 2I\otimes I\right]\frac{C}{2}. \tag{B-12}$$

Combining this expression with equation set (63), and then using equations (A-8) and (A-23) along with some algebraic manipulations, yields the result

$$\underline{\xi}_4(\underline{x};\underline{\hat{x}},\Gamma) = \left[\frac{C}{2}\otimes\frac{C}{2}\right]\left[\Gamma^{1/2}\otimes\ldots\right]^4 vec\{\Omega(\underline{z})\}, \tag{B-13}$$

where

$$\Omega(\underline{z}) = \left\{vec\left(\underline{zz}^T-\Gamma^{-1}\right)\right\}\left\{vec\left(\underline{zz}^T-\Gamma^{-1}\right)\right\}^T - 4\Gamma^{-1}\otimes\left(\underline{zz}^T\right) + 2\Gamma^{-1}\otimes\Gamma^{-1}, \tag{B-14}$$

and $\underline{z}$ is defined by equation (B-6). With the aid of equations (B-13) and (65e), and the identity

$$\underline{\beta}_4^T\left[\frac{C}{2}\otimes\frac{C}{2}\right] = \underline{\beta}_4^T, \tag{B-15}$$

it follows that

$$\underline{\beta}_4^T \underline{\xi}_4(\underline{x};\hat{\underline{x}},\Gamma) = \Big[\int_{R_n}\big[(\underline{\tilde{x}}-\hat{\underline{x}})\otimes\ldots\big]^4 p(\underline{\tilde{x}})d\underline{\tilde{x}} - vec\{C(\Gamma\otimes\Gamma)\}$$

$$- vec(\Gamma)\otimes vec(\Gamma)\Big]^T vec\{\Omega(\underline{z})\}. \tag{B-16}$$

Next, observe the relation

$$\left[vec\{C(\Gamma\otimes\Gamma)\}\right]^T vec\{\Omega(\underline{z})\} = 2\left[vec(\Gamma)\otimes vec(\Gamma)\right]^T vec\{\Omega(z)\}. \tag{B-17}$$

If

$$\Delta_4 = \int_{R_n}\left[(\underline{\tilde{x}}-\hat{\underline{x}})(\underline{\tilde{x}}-\hat{\underline{x}})^T\right]\otimes\left[(\underline{\tilde{x}}-\hat{\underline{x}})(\underline{\tilde{x}}-\hat{\underline{x}})^T\right]p(\underline{\tilde{x}})d\underline{\tilde{x}} \tag{B-18}$$

is defined as the matrix of fourth order central moments associated with $p(\underline{x})$, then equations (B-17), (B-18), and (A-26) allow equation (B-16) to be rewritten in the desired form; i.e.,

$$\underline{\beta}_4^T \underline{\xi}_4(\underline{x};\hat{\underline{x}},\Gamma) = tr\left\{\left[\Delta_4 - 3\{vec(\Gamma)\}\{vec(\Gamma)\}^T\right]\Omega(z)\right\}. \tag{B-19}$$

Equations (B-10) and (B-19) represent simplified expressions for the first two non-trivial terms in the Gram-Charlier series expansion of $p(\underline{x})$. It is noteworthy that analogous expressions also exist for the corresponding non-trivial terms in the likelihood ratio representation given by equation (94). They are

$$\underline{\beta}_3^T\left[\left(\Gamma_s^{1/2}\left[\Gamma_{\underline{s}}+\Gamma_n\right]^{1/2}\right)\otimes\ldots\right]^3 \underline{\xi}_3\left(\underline{r};\{\hat{\underline{s}}+\hat{\underline{n}}\},\left[\Gamma_{\underline{s}}+\Gamma_{\underline{n}}\right]\right)$$

$$= tr\left\{\Delta_3^*\left[\left(\underline{yy}^T - 3\left[\Gamma_{\underline{s}}+\Gamma_{\underline{n}}\right]^{-1}\right)\otimes\underline{y}\right]\right\}, \tag{B-20}$$

$$\underline{\beta}_4^T\left[\left(\Gamma_{\underline{s}}^{1/2}\left[\Gamma_{\underline{s}}+\Gamma_{\underline{n}}\right]^{1/2}\right)\otimes\ldots\right]^4 \underline{\xi}_4\left(\underline{r};\{\hat{\underline{s}}+\hat{\underline{n}}\},\left[\Gamma_{\underline{s}}+\Gamma_{\underline{n}}\right]\right)$$

$$= tr\left\{\left[\Delta_4^* - 3\{vec(\Gamma_{\underline{s}})\}\{vec(\Gamma_{\underline{s}})\}^T\right]\Omega^*(\underline{y})\right\}, \tag{B-21}$$

where

$$\underline{y} = \left[\Gamma_{\underline{x}}+\Gamma_{\underline{n}}\right]^{-1}(\underline{r}-\hat{\underline{s}}-\hat{\underline{n}}), \tag{B-22}$$

$$\Delta_3^* = \int_{R_n} \left[ (\underline{s} - \hat{\underline{s}})(\underline{s} - \hat{\underline{s}})^T \right] \otimes (\underline{s} - \hat{\underline{s}})^T p_{\underline{s}}(\underline{s}) d\underline{s}, \tag{B-23}$$

$$\Delta_4^* = \int_{R_n} \left[ (\underline{s} - \hat{\underline{s}})(\underline{s} - \hat{\underline{s}})^T \right] \otimes \left[ (\underline{s} - \hat{\underline{s}})(\underline{s} - \hat{\underline{s}})^T \right] p_{\underline{s}}(\underline{s}) d\underline{s}, \tag{B-24}$$

and

$$\begin{aligned} \Omega^*(\underline{y}) = & \left\{ vec\left( \underline{yy}^T - \left[ \Gamma_{\underline{s}} + \Gamma_{\underline{n}} \right]^{-1} \right) \right\} \left\{ vec\left( \underline{yy}^T - \left[ \Gamma_{\underline{s}} + \Gamma_{\underline{n}} \right]^{-1} \right) \right\}^T \\ & - 4\left[ \Gamma_{\underline{s}} + \Gamma_{\underline{n}} \right]^{-1} \otimes \left( \underline{yy}^T \right) + 2\left[ \Gamma_{\underline{s}} + \Gamma_{\underline{n}} \right]^{-1} \otimes \left[ \Gamma_{\underline{s}} + \Gamma_{\underline{n}} \right]^{-1}. \end{aligned} \tag{B-25}$$

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